Stochastic Modelling of Sieving

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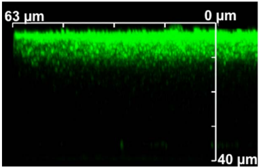
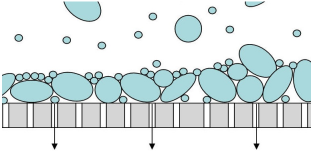
Outline

Fouling Mechanisms

- Adsorption: Deposition of **small** particles on the pore walls within membrane;
- Sieving: Deposition of **large** particles covering the top of membrane or plugging pores;
- Cake formation: Stacking of particles on top of each other, during later stages of filtration.





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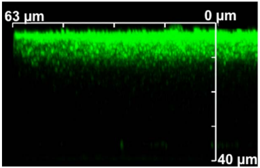

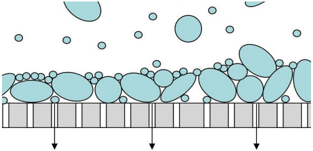

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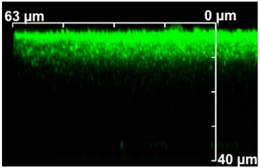

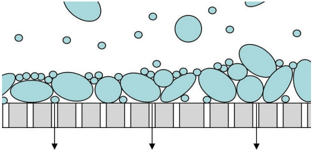

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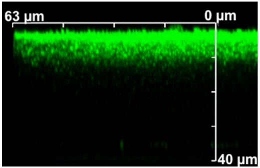
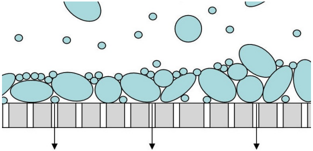
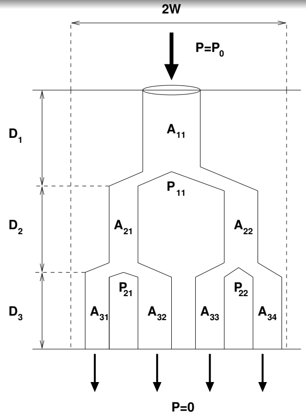




Flow and Adsorption (Sanaei & Cummings (2017))



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Hagen Poiseuille

$$\pi A_{ij}^2 \overline{U}_{\mathrm{p},ij} = -\frac{1}{\mu R_{ij}} \frac{\Delta P}{D_i}.$$

Conservation of Flux

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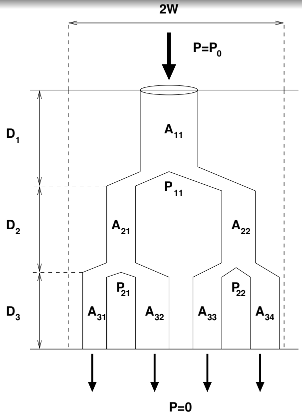

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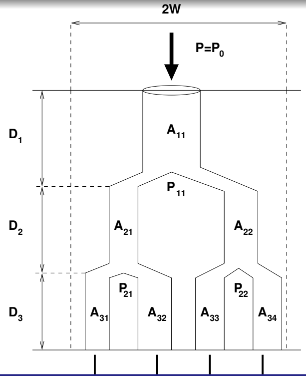

Advective Deposition & Pore Shrinkage

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$$\frac{\partial A_{ij}}{\partial T} = -\Lambda \alpha C_{ij},\ A_{ij,0} = A_{ij}(0), \quad 1 \le i \le m, \quad 1 \le j \le \nu_i.$$

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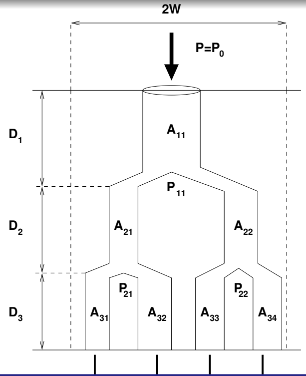

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Discrete Properties

- Takes one large particle to have an effect
- View the underlying geometry as a connected graph.

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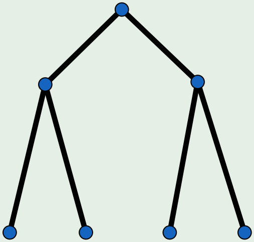
A connected graph



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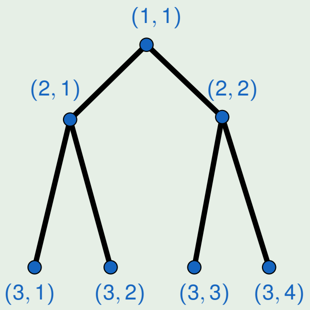

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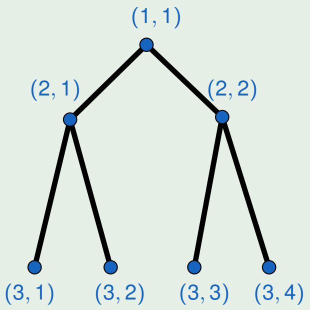

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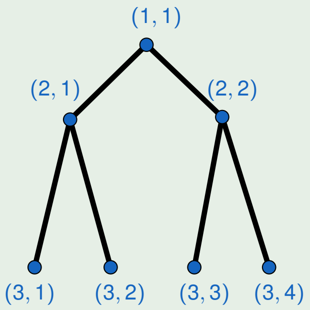

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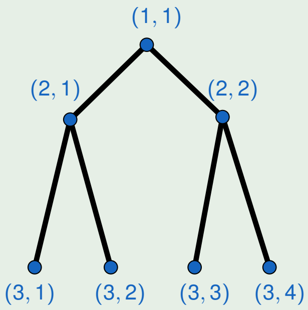

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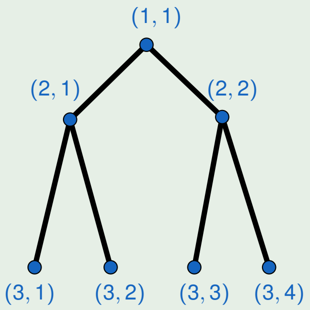

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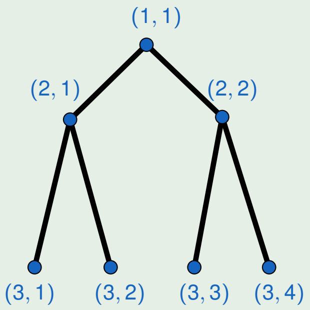

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- Determinantion of blocking state by size comparison between particle and pore size

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- The size and arrival of particles are independent.

Effect of Local Sieving

State Characterization

$$\chi_{ij}(T) = \begin{cases} 1, & T \geq T_{ij} \text{ (blocked)}, \\ 0, & T < T_{ij} \text{ (open)}, \end{cases}$$

where T_{ij} is the blocking time of a pore at location (i, j).

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Local Resistance due to Sieving

$$R_{ij,\text{blocking}} = \Xi \mathbb{E}\left[\chi_{ij}\right]$$

Ξ: Severity of sieving.

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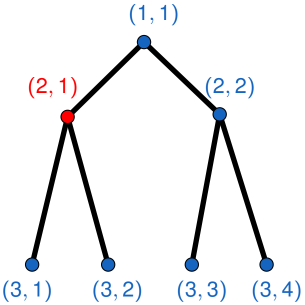
Solution: condition on the state of the ensemble.

Continuous-time Markov Chain (CTMC)



$(0, 0, 1, 0, 0, 0, 0, 0)$

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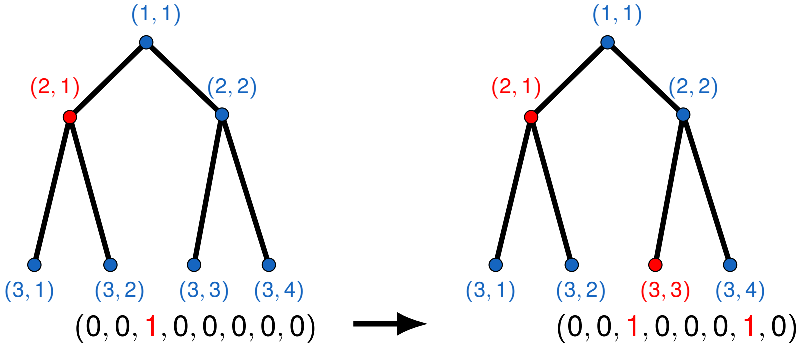

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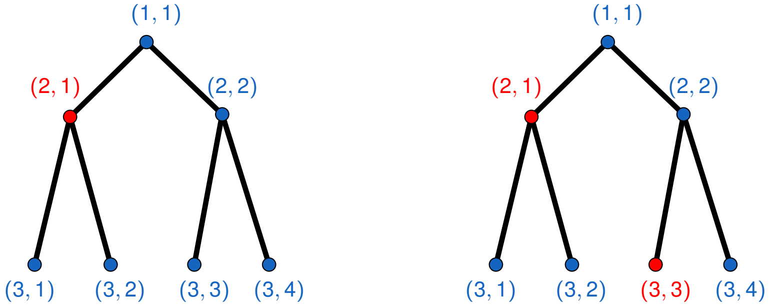

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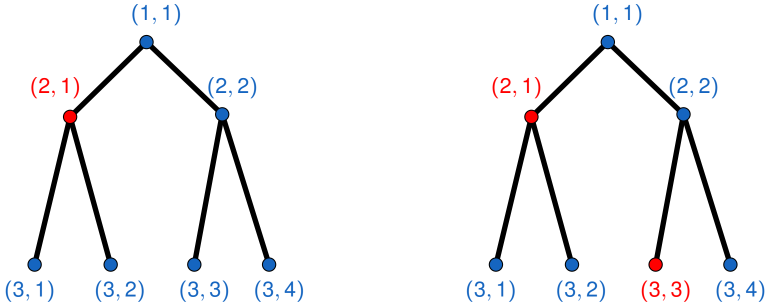

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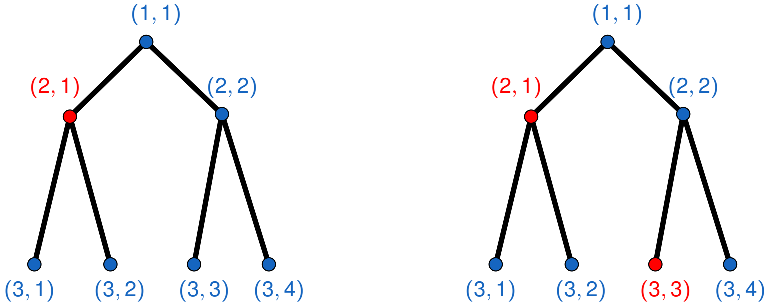

$Z_T = (0, 0, 1, 0, 0, 0, 0, 0) \longrightarrow Z_{T+dT} = (0, 0, 1, 0, 0, 0, 1, 0)$

Z_T : time-inhomogeneous CTMC with transition density $\mathcal{P}$ and rate $\mathcal{Q}$, satisfying the master equations,

$$\frac{d\vec{P}}{dT} = \mathcal{Q}(T)\,\vec{P}(T)\,,\, \vec{P}(0) = (1, 0, \ldots, 0)\,.$$

$\vec{P}$ lists the probability of the ensemble being in a possible state, contributing to a new formula for $\mathbb{E}\left[T_{ij}\right]$.

Resistance in Series

- Resistance by the two fouling modes are put in series,

$$R_{ij} = R_{ij,a} + R_{ij,b} = R_{ij_a} = \frac{8}{\pi A_{ij}^4} + \Xi \mathbb{E}\left[T_{ij}\right].$$

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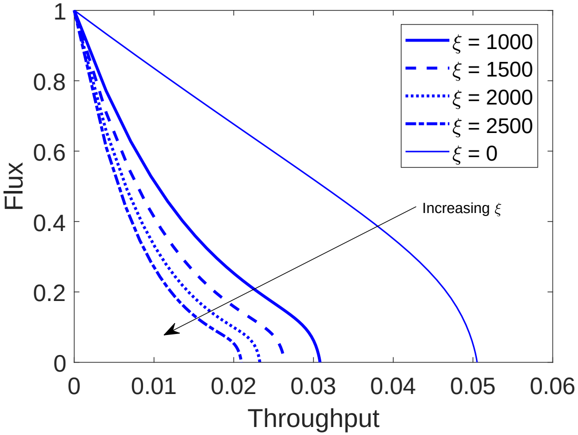
- Allow leakage after sieving has occurred.
- Nondimensionalise.
- Fix initial resistance for fair comparison of different membranes.
- Initialize pore size (for symmetric case – pores in the same layer have equal initial radii):

$$a_i(0) = a_0 \kappa^{i-1}.$$

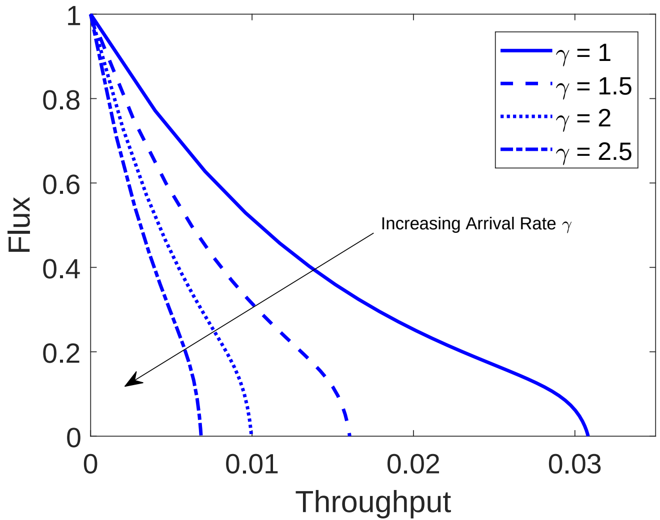
Variations in Sieving Severity ξ

Throughput:

$$v(t) = \int_0^t u(s)\, ds.$$



Variations in Arrival Rates γ



Model Predictions and Extensions

- Increasing blocking strength and arrival rate reduces total throughput, as expected.
- Sieving severity parameter can be further improved.
    - Radius dependent, e.g. sieving impact is small when pore radius is small.
- Apply the method to more general connected graphs.
- Investigate the dependence of membrane performance on graph parameters.

Acknowledgments

- “Flow and fouling in membrane filters: effects of membrane morphology”, Sanaei, P. & Cummings, L. J., Journal of Fluid Mechanics, 818, 744–771 (2017).
- “Membrane filtration with complex branching pore morphology”, Sanaei, P. & Cummings, L. J., Physical Review Fluids, 3(9), 094305 (2018)."

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