Cleaning flows with hydraulic jumps: thin film approximation and beyond

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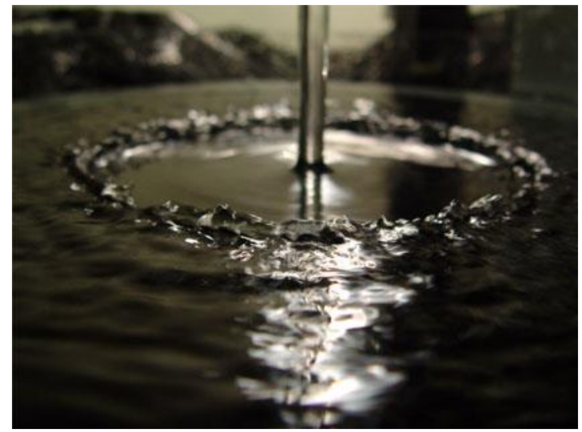
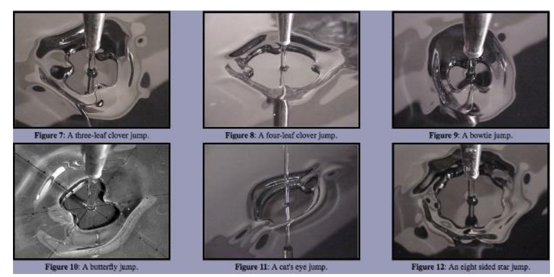
Hydraulic jumps

Nearly laminar and circular symmetric ⟷ **Turbulent and asymmetric**





Hydraulic jumps: features

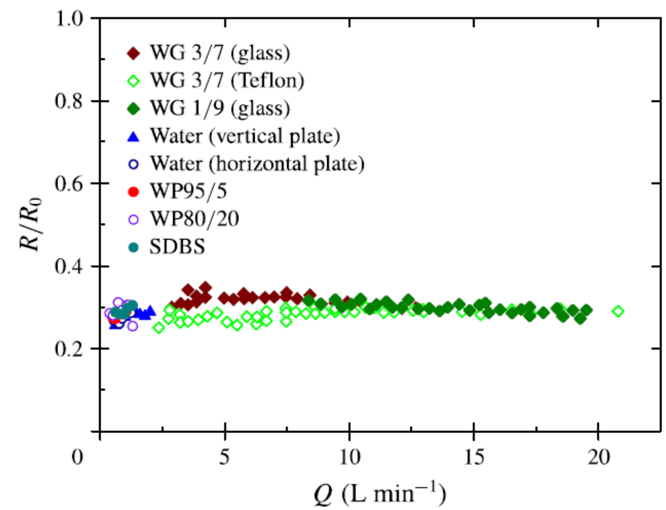
Far field conditions and the jump position

Flow out

Parametric dependencies and the mechanism – viscous μ and surface tension γ forces

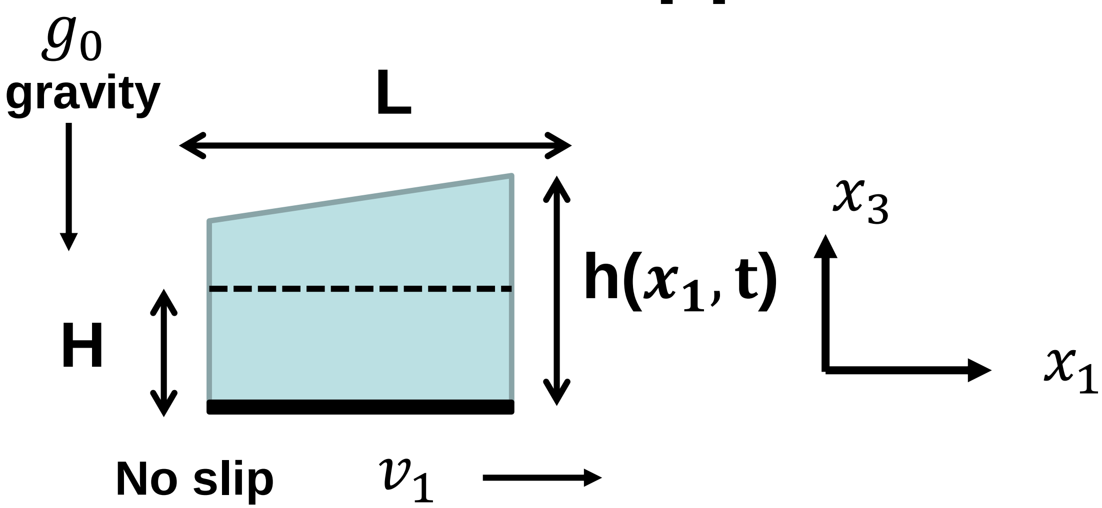
Jump position R/R_0 as a function of the flow rate Q

Bhagat, Jha, Linden & Wilson 2018



$$R_0 = \frac{Q^{3/4}\rho^{1/2}}{\mu^{1/4}\gamma^{1/4}}$$

Thin film approximation (1D Cartesian)



$$\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{x_1}\left\{\frac{q^2}{h}\right\} = -h\frac{\partial p}{\partial x_1} - \frac{3}{\widehat{Re}}\frac{q}{h^2}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x_1} = 0$$

$$p = p_0 + Ka(h - x_3) - \frac{1}{\widehat{Re}\widehat{Ca}}\left(\frac{\partial^2 h}{\partial x_1^2}\right)$$

- $\delta = \frac{H}{L} \ll 1$
- $Re \gg 1$
- $\widehat{Re} = \delta^2 Re$
- $q = \int_0^h v_1 dx_3$
- $Ka = \delta\frac{g_0 L}{U^2} \sim O(1)$
- $\widehat{Ca} = \frac{Ca}{\delta^3}$
- $\widehat{Re}\widehat{Ca} \sim O(1)$
- $Re = \frac{\rho U L}{\mu}$ **Reynolds number**
- $Ca = \frac{\mu U}{\gamma}$ **Capillary number**

Thin film approximation (1D Radial)

$$\frac{\partial q}{\partial t} + \frac{6}{5}\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{rq^2}{h}\right\} = -h\frac{\partial p}{\partial r} - \frac{3}{\widehat{Re}}\frac{q}{h^2}$$

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial qr}{\partial r} = 0$$

$$p = p_0 + Ka(h - x_3) - \frac{1}{\widehat{Re}\widehat{Ca}}\left(\frac{\partial^2 h}{\partial r^2}\right)$$

Thin film approximation: features

Pros

- No need to track free surface, it is part of the solution
- Easier to simulate and interpret

Cons

- Eddy (turbulent) motion can not be described in principle

What can we obtain:

- A good benchmark test for comparison
- Some ideas about parameter scaling
- Nature of the jump

Thin film approximation: scaling

Empirical ($We = \frac{\rho U_h^2 h}{\gamma} \sim 1, \frac{h}{R_0}\frac{\rho U_h h}{\mu} \sim 1$) scaling (Bhagat, Jha, Linden & Wilson 2018) in the radial case in terms of dimensional parameters

$$R_0 = \frac{Q^{3/4}\rho^{1/2}}{\mu^{1/4}\gamma^{1/4}}$$

Experimental scaling in terms of non-dimensional parameters of the thin film approximation, radial case at Ka<<1

$$\frac{R_0}{R_c} = \widehat{Re}^{1/2}\widehat{Ca}^{1/4}\delta^{1/2}$$

Parameter δ does not explicitly contribute into the thin film equations

Typical parameter range in Bhagat, Jha, Linden & Wilson 2018

$$\widehat{Re} = 30 \gg 1, \widehat{Ca} = 3$$

$$Ka = 0.1 \ll 1$$

gravity is not essential?

Thin film approximation: scaling

In a general case

$$\frac{1}{We} + \frac{1}{Fr^2} = 1 \qquad We = \frac{\rho U_h^2 h}{\gamma}, Fr = \frac{U_h^2}{g_0 h}$$

We – Weber number
Fr – Froude number

$$\frac{R_0}{R_c} = \left(\frac{\sqrt{1 + 4\widehat{Ca}^2\,\widehat{Re}\,Ka\,\delta^4} - 1}{2\,\dfrac{\widehat{Ca}\,Ka\,\delta^2}{\widehat{Re}}} \right)^{1/4}$$

$Ka \ll 1$	$Ka \gg 1$
$$\frac{R_0}{R_c} = \widehat{Re}^{1/2}\widehat{Ca}^{1/4}\delta^{1/2}$$	$$\frac{R_0}{R_c} = \frac{\widehat{Re}^{3/8}}{Ka^{1/8}}$$

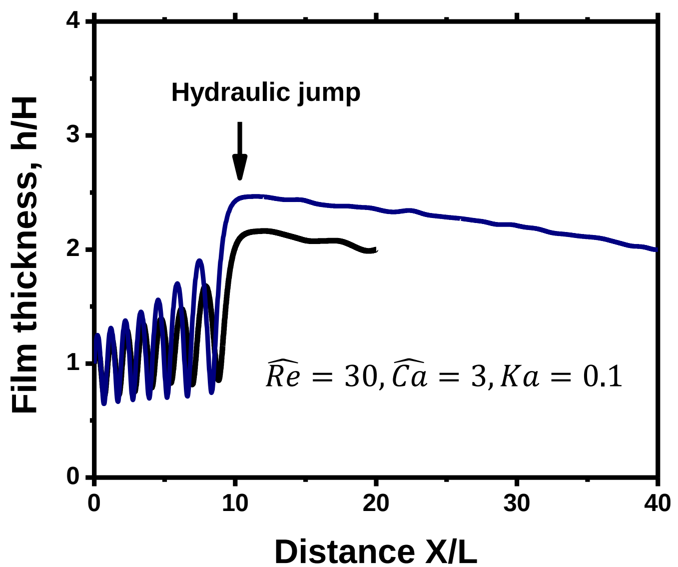
In a 1D Cartesian case at Ka<<1

$$\frac{X_0}{L} = \widehat{Re}^2\widehat{Ca}\,\delta^2$$

Weak solutions to the thin film equations: a boundary value problem in 1D Cartesian case

System of the thin film equations has
a unique, regular weak solution
subject to regular initial data

$$q(0,t) = 1$$
$$h(0,t) = 1$$

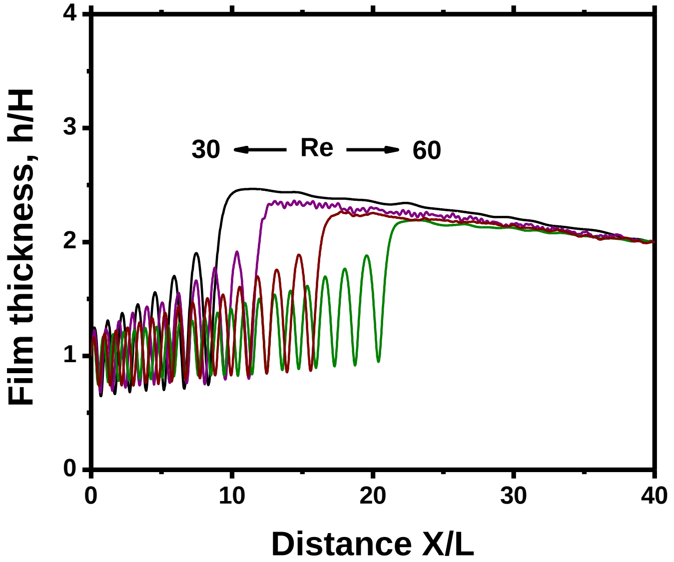


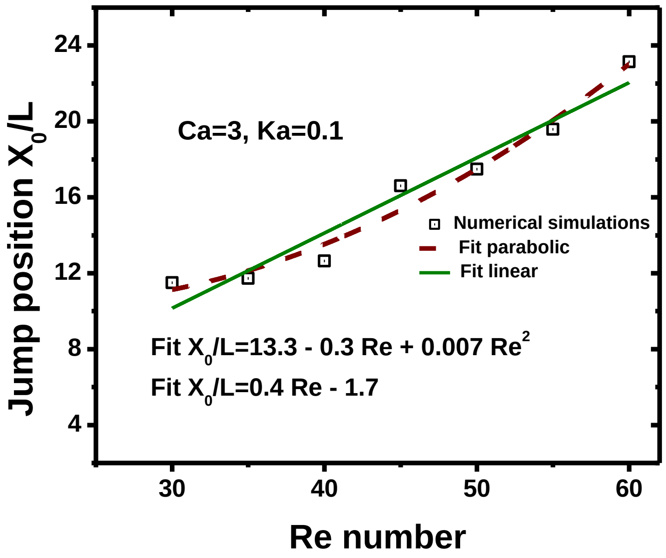
$$h(X,t) = h_e$$
$$h_x(X,t) = 0$$

Boundary value problem, 1D Cartesian steady state

Parametric dependencies (Re)

1D Cartesian steady states

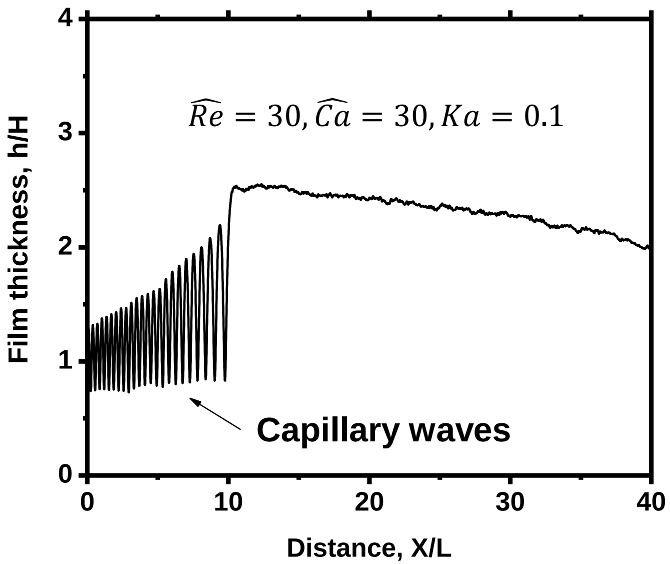


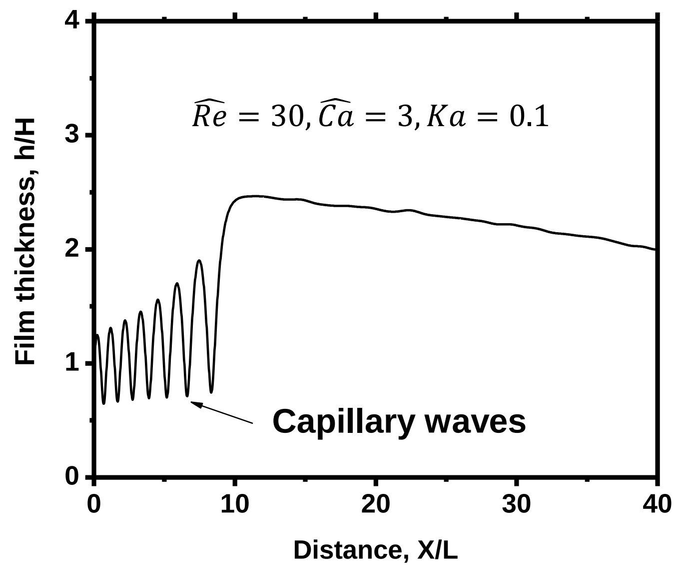


$$X_0 \propto \widehat{Re}$$

Parametric dependencies ($\widehat{Ca}$)

1D Cartesian steady states

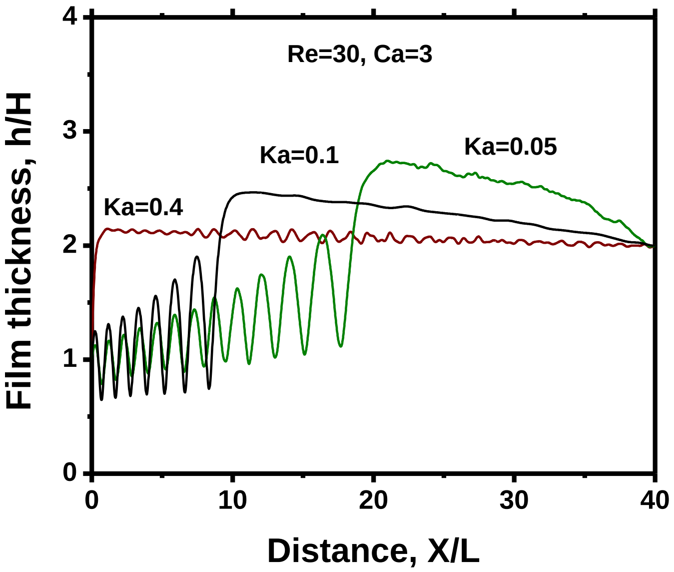


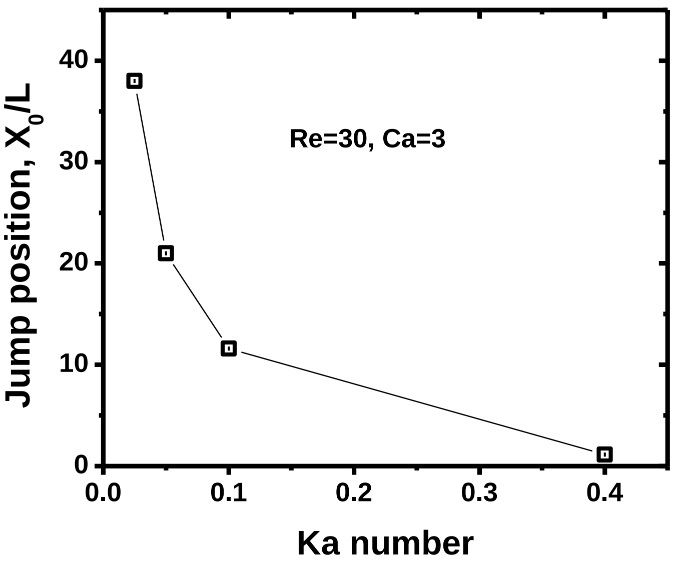


Capillary wave length λ is affected by Ca number variations in the normal way $\lambda \propto \widehat{Ca}^{-1/2}$, but the jump position is practically indifferent to $\widehat{Ca}$ number.

Parametric dependencies (Ka)

1D Cartesian steady states





Conclusions

- ❑ In the thin film approximation, there is a regular, weak solution of a boundary value problem featuring the existence of a hydraulic jump
- ❑ In1D Cartesian geometry simulations, the hydraulic jump position in a steady state is independent of the position of the far field boundary
- ❑ The appearance of the hydraulic jump in1D Cartesian geometry seems to be quite sensitive to the Ka number, the non-dimensional number representing the role of **gravity** in the system
- ❑ At the same time, the jump position is **indifferent to the capillary action**, in particular to the value of the Ca number, while being **linearly dependent on the Re number**
- ❑ **The behaviour of the hydraulic jump in 1D Cartesian geometry is quite different from that in a radially symmetric case, if we trust the model (thin film approximation) and the numerical method**